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# GRACEFULNESS IN THE PATH UNION OF VERTEX SWITCHING OF EVEN CYCLES IN INCREASING ORDER

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**Abstract:** A *graceful labeling* of a graph $G$ with $q$ edges is an injection $f : V(G) \rightarrow \{0, 1, 2, ..., q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with the vertices $u$ and $v$ is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph. In this paper, we prove that the path union of vertex switching of even cycles in increasing order is graceful.

**Keywords and Phrases:** Graceful labeling, vertex switching, Path union of graphs.

**2010 Mathematics Subject Classification:** 05C78.

## 1. Introduction

The most famous and challenging graph labeling method is the graceful labeling of graphs introduced by Rosa [9] in 1967. A *graceful labeling* of a graph $G$ with $q$ edges is an injection $f : V(G) \rightarrow \{0, 1, 2, ..., q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with the vertices $u$ and $v$ is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a *graceful graph*. A variety of graphs and families of graphs are known to be graceful for the past five decades. Caterpillars are proved to be graceful by Rosa [9]. Morgan [8] has shown that all lobsters with perfect matchings are graceful. Hrnciar and Haviar [6] have shown that all trees of diameter five are graceful.

Golomb [3] has proved that the complete bipartite graph $K_{m,n}$ is graceful. Rosa [9] showed that the $n$-cycle $C_n$ is graceful if and only if $n \equiv 0$ or 3 ($mod$ 4). Wheels $W_n = C_n + K_1$ are shown to be graceful [5]. Helms are shown to be graceful [1]. Vaidya *et.al*. [10] have defined a vertex switching $G_v$ as the graph obtained from $G$ by removing all edges incident to $v$ and adding edges joining $v$ to every vertex not adjacent to $v$ in $G$. Let $G_1, G_2, G_3, ..., G_n$ be $n \geq 2$ copies of a graph $G$. Then the graph $G(n)$ obtained by adding an edge to $G_i$ and $G_{i+1}$, $i = 1, 2, ..., (n-1)$ is called the path-union of $n$ copies of the graph $G$ [7]. Kaneria *et.al*. [7] have proved that the path union of complete bipartite graphs is graceful. For an exhaustive survey on graceful graphs refer to the dynamic survey by Gallian [2]. Ghodasara *et.al*. [4] have defined the path union of vertex switching of cycle graphs which he proved that it is cordial. In this paper, we prove that the path union of vertex switching of even cycles in increasing order is graceful.

## 2. Main Results

In this section we first recall the definition for cycle graph, vertex switching of a graph, path union of graphs. Later we prove that the path union of vertex switching of even cycles in increasing order is graceful.

**Definition 2.1.** *A sequence of vertices* $[v_0, v_1, v_2, ..., v_n, v_0]$ *is a cycle of length* $n+1$ *if* $v_{i+1}v_i \ \varepsilon \ E$, $i = 1, 2, 3, ..., n$ *and* $v_n v_0 \ \varepsilon \ E$. *A cycle of length* $n$ *is denoted by* $C_n$.

**Definition 2.2.** *A vertex switching[10]* $G_v$ *of a graph* $G$ *is the graph obtained by taking a vertex* $v$ *of* $G$*, removing all the edges incident to* $v$ *and adding edges joining* $v$ *to every other vertex which are not adjacent to* $v$ *in* $G$.

**Definition 2.3.** *Let* $G$ *be a graph and* $G_1, G_2, ..., G_n$ $(n \geq 2)$ *be* $n$ *copies of graph* $G$. *Then the graph obtained by adding an edge from* $G_i$ *to* $G_{i+1}$, $i = 1, 2, 3, ..., (n-1)$ *is called the path union of* $G$.

**Theorem 2.4.** *The path union of vertex switching of even cycles in increasing order is graceful.*

**Proof.** Let $G$ be a cycle $C_n$ with $n$ vertices that are denoted as $v_1, v_2, ..., v_n$ in the anticlockwise direction. Let $H$ be the vertex switching graph of the graph $G$ with $v_1 \ \varepsilon \ G$ as the switching vertex. The graphs $G$ and $H$ are shown in Figure 1.

Let $H_1, H_2, ..., H_m$ be the copies of $H$ in an increasing order as shown in Figure 2. The first copy $H_1$ of $H$ is described as follows. Denote the switching vertex of $H_1$ as $v_1^1$. Denote the remaining vertices in $H_1$ as $v_2^1, v_3^1, ..., v_6^1$ in the anticlockwise direction. The second copy $H_2$ of $H$ is described as follows. Denote the switching vertex of $H_2$ as $v_1^2$. Denote the remaining vertices in $H_2$ as $v_2^2, v_3^2, ..., v_8^2$ in the anticlockwise direction. Finally the last copy $H_m$ of $H$ is described by denoting the switching vertex as $v_1^m$. The remaining vertices of $H_m$ are denoted as $v_2^m, v_3^m, ..., v_k^m$ where $k = n_i$ for $1 \leq i \leq m$ in the anticlockwise direction.

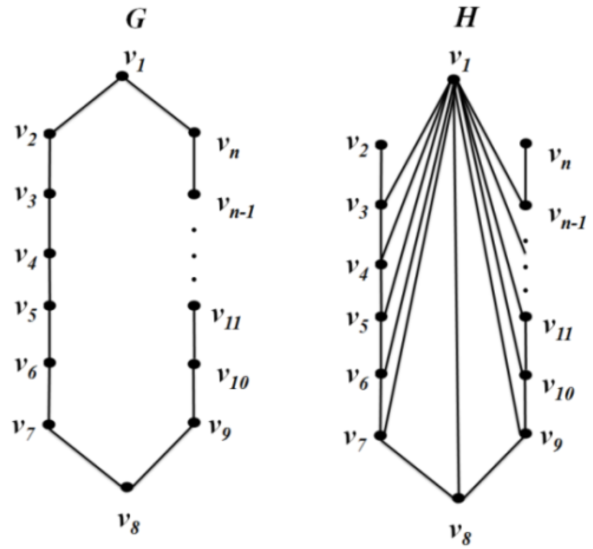


**Figure 1: The graphs $G$ and $H$**

Let $H^{'}$ be the graph obtained by adding an edge $e_i$ between the switching vertices $v_1^i$ and $v_1^{i+1}$ of the copies $H_i$ and $H_{i+1}$ ,$1 \leq i \leq (m-1)$. The graph $H^{'}$ so obtained is called the path union of vertex switching of even cycles as shown in the Figure 2.

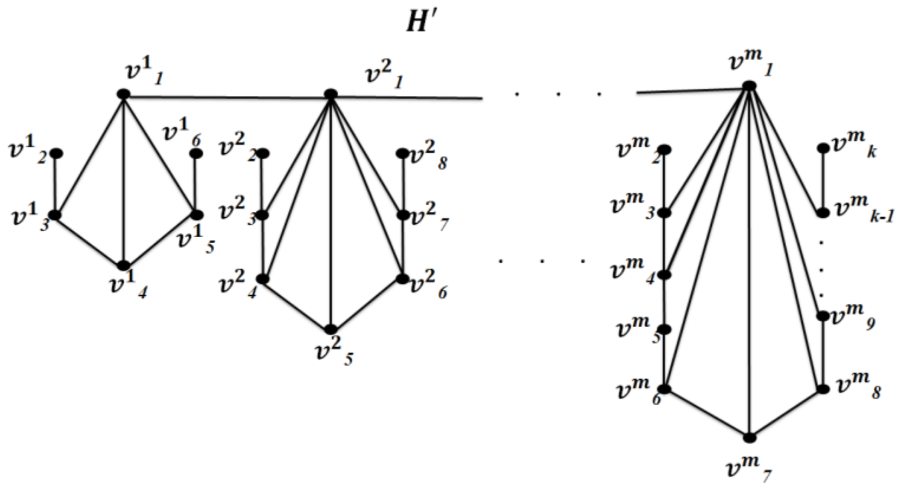


**Figure 2: The graph $H^{'}$ which is the path union of the vertex switching of even cycles in increasing order**

Note that in $H'$ the switching vertices are $v_1^i$ for $1 \leq i \leq m$ and the remaining vertices are $v_j^i$ for $(1 \leq i \leq m)$,$(2 \leq j \leq n_i)$. If $p$ denotes number of vertices in $H'$ then $p = \sum_{i=1}^{m} n_i$ and if $q$ denotes the number of edges in $H'$ then $q = \sum_{i=1}^{m} n_i + m^2 + (m-1)$. Also note that the theorem is proved for $(n_i \equiv 0(mod\ \ 2)$ and $n_i \geq 6)$ The vertices of $H'$ are labeled as follows depending on the parameter $m$.

Labels for the switching vertices $v_1^i$ are given below for $(1 \leq i \leq m)$

$$\begin{array}{ll}
f(v_1^1) = q & \\
f(v_1^{2i+1}) = q - \sum_{k=1}^{i}(\frac{n_{2k}}{2}) & for \quad 1 \leq i \leq \lfloor \frac{m}{2} \rfloor \\
f(v_1^2) = (\frac{n_1}{2} - 1) & \\
f(v_1^{2i}) = (\frac{n_1}{2} - 1) + \sum_{k=1}^{i-1}(\frac{n_{2k+1}}{2}) & for \quad 2 \leq i \leq \lfloor \frac{m}{2} \rfloor
\end{array}$$

Labels for the vertices $v_{2j}^i$ for $1 \leq i \leq m$, $1 \leq j \leq n_i$ are given below

**Case 1:** $m \equiv 1(mod\ 2)$

$$\begin{array}{lll}
f(v_{2j}^1) = \lfloor \frac{q}{2} \rfloor - (j-1) & for & 1 \leq j < \frac{n_1}{2} \\
f(v_{2j}^1) = \lfloor \frac{q}{2} \rfloor + (j-1) & for & j = \frac{n_1}{2} \\
f(v_{2j}^{2i+1}) = \lfloor \frac{q}{2} \rfloor - (3i^2 + 4i) & & \\
\qquad -(j-1) & for & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor, 1 \leq j < \frac{n_{2i+1}}{2} \\
f(v_{2j}^{2i+1}) = \lfloor \frac{q}{2} \rfloor - (3i^2 + 4i) & & \\
\qquad +(j-1) & for & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor, j = \frac{n_{2i+1}}{2}
\end{array}$$

**Case 2:** $m \equiv 0(mod\ 2)$

$$\begin{array}{lll}
f(v_{2j}^2) = \lfloor \frac{q}{2} \rfloor + 4 + (j-1) & for & 1 \leq j < \frac{n_2}{2} \\
f(v_{2j}^2) = \lfloor \frac{q}{2} \rfloor + 4 - (j-1) & for & j = \frac{n_2}{2} \\
f(v_{2j}^{2i}) = \lfloor \frac{q}{2} \rfloor + 4 + \sum_{k=1}^{i-1}[(\frac{n_{2k}+n_{2k+1}}{2}) & & \\
\qquad +(2k-1)] + (j-1) & for & 2 \leq i \leq \lfloor \frac{m}{2} \rfloor, 1 \leq j < \frac{n_{2i}}{2} \\
f(v_{2j}^{2i}) = \lfloor \frac{q}{2} \rfloor + 4 + \sum_{k=1}^{i-1}[(\frac{n_{2k}+n_{2k+1}}{2}) & & \\
\qquad +(2k-1)] - (j-1) & for & 2 \leq i \leq \lfloor \frac{m}{2} \rfloor, j = \frac{n_{2i}}{2}
\end{array}$$

Labels for the vertices $v_{2j+1}^i$ for $1 \leq i \leq m$, $1 \leq j \leq n_i$ are given below

**Case 1:** $m \equiv 1(mod\ 2)$

$$\begin{array}{lll}
f(v_{2j+1}^1) = (j-1) & for & 1 \leq j < \frac{n_1}{2} \\
f(v_{2j+1}^{2i-1}) = \sum_{k=1}^{i-1}(\frac{n_{2k-1}}{2} + (j-1) & for & 2 \leq i \leq \lceil \frac{m}{2} \rceil, 1 \leq j < \frac{n_{2i-1}}{2}
\end{array}$$

**Case 2:** $m \equiv 0(mod\ 2)$

$$f(v^2_{2j+1}) = q - j \qquad for \qquad 1 \leq j < \frac{n_2}{2}$$
$$f(v^{2i-1}_{2j+1}) = q - \sum_{k=1}^{i-1} \frac{n_{2k}}{2} - j \quad for \quad 2 \leq i \leq \lceil \frac{m}{2} \rceil, 1 \leq j < \frac{n_{2i}}{2}$$

From the above definition it is clear that all the vertex labels are distinct. The edge labels can be computed from the above vertex labels and are also found to be distinct from 1 to $q$. Therefore, the path union of copies of the vertex switching of even cycles in increasing order is graceful for ($n_i \equiv 0(mod\ \ 2)$ and $n_i \geq 6$). The theorem is illustrated in Figure 3.

**Illustration:** $q = 79, p = 50, m = 5, n_1 = 6, n_2 = 8, n_3 = 10, n_4 = 12, n_5 = 14$

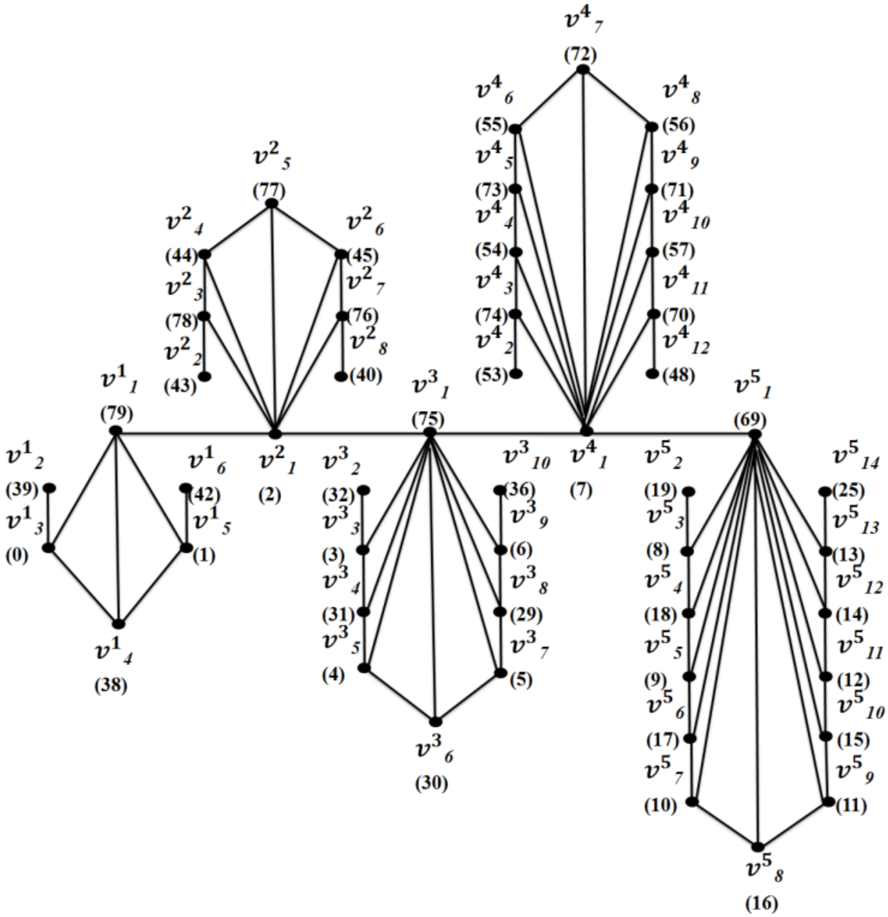


**Figure 3: Path union of the vertex switching of even cycles in increasing order is graceful when m=5**

## 3. Conclusion

In this paper we have proved that the path union of copies of the vertex switching of even cycles in increasing order is graceful where for ($n_i \equiv 0(mod \ \ 2)$ and $n_i \geq 6$). Further we intend to prove the gracefulness of path union of vertex switching of odd cycles in increasing order.

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